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ON THE LUNAR AND PLANETARY THEORIES.

BY JOHN N. STOCKWELL.

In my Theory of the Moon's Motion, which was published about a year ago, I have called the attention of astronomers to a class of inequalities which appear to have been incorrectly calculated by all my predecessors in this interesting field of inquiry. The inequalities to which I refer, are of great importance in the lunar theory, not only on account of their own magnitude, but also on account of their modifying influence on some of the other inequalities of the moon's motion. They are wholly independent of the coordinates of the disturbing body; and the coefficients of the time in the arguments of the equations differ from unity but by quantities of the order of the disturbing forces. They arise from the integration of equations of the form,

$$\frac{d^2\delta r}{dt^2} + N^2\delta r + m^2\cos(it-\varepsilon) = 0; \tag{1}$$

in which N and i differ from unity but by quantities of the order m^2, which here represents the disturbing function.

The general integral of equation (1) is

$$\delta r = \frac{m^2}{i^2-N^2}\cos(it-\varepsilon). \tag{2}$$

For the particular inequality to which I wish to call attention in this paper, equation (1) becomes

$$\frac{d^2\delta\frac{1}{r}}{dt^2} + N^2\delta\frac{1}{r} - \tfrac{15}{4}m^2e\gamma^2\cos(nt+\omega-2\Omega) = 0; \tag{3}$$

the development being in accordance with Ponte coulant's work on the lunar theory, and the notation being the same as in my own work; namely, r denotes the radius vector; e and γ, the eccentricity and inclination of the moon's orbit; nt, ω and Ω, the mean longitude of the moon, the perigee and the node, respectively.

By reason of the perturbations ω and Ω are variable, and we have

$$\delta\omega = -\delta\Omega = \tfrac{3}{4}m^2nt. \tag{4}$$

The coefficient of nt in the argument of equation (3) therefore becomes $1+\frac{9}{4}m^2 = i$; while N is equal to uuity minus the motion of the perigee, or $N = 1 - \frac{3}{4}m^2$. Therefore

$$i^2 - N^2 = 6m^2, \tag{5}$$

and equation (3) gives by integration,

$$\delta\frac{1}{r} = -\tfrac{5}{8}e\gamma^2\cos(nt + \omega - 2\Omega). \tag{6}$$

The inequality depending on this argument is, therefore, apparently independent of the disturbing force, since the disturbing function becomes a factor of both numerator and denominator of the integral expressed by equation (2). La Place designates such inequalities as *finite.* They are, however, really indeterminate by equation (2), as we shall now proceed to show. It is evident that equation (1) becomes more accurate the smaller the disturbing function m^2 becomes, since it is the first term of a converging series depending on the ascending powers of m^2. Equation (1) would therefore be true if m^2 were an infinitesimal; but equation (2) gives the same value of δr whether m^2 be infinitesimal or finite; a result which is in direct contravention of the general principle of mechanics that the effect is proporportional to the cause.

But at this point we are met by the objection of Mr. G. W. Hill, who remarks, in the March No. of the ANALYST, that the value "$m=0$ implies that we have either an infinitely short month or an infinitely long year"; and is therefore not an admissible hypothesis, since the value of m is really finite.

Now Mr. Hill's remark is correct only in the case in which the sun is the disturbing body; and does not apply to the case of a second satellite or a planet. In this latter case, if we put the constant term of the expression of the inverse cube of the planet's distance from the earth equal to $b \div a''^3$, a'' being the mean distance of the planet from the sun; and also put

$$\mathrm{m}^2 = m''b\,\frac{a^3}{a''^3}, \tag{7}$$

m'' being the planet's mass; we shall have

$$\delta\omega = -\delta\Omega = \tfrac{3}{4}\mathrm{m}^2 nt \qquad \text{and} \qquad i^2 - N^2 = 6\mathrm{m}^2;$$

but epuation (6) would remain unchanged.

In the case of Neptune being the disturbing body, we shall find $\mathrm{m}^2 = \frac{1}{9000000000000}$; and this value of the disturbing force would cause the perigee and node of the moon's orbit to move at the rate of about one second of arc in a period of 7000 years; but the substitution of these quantities in equation (3) gives the same perturbation as the disturbing force of the sun, which is about 500,000,000 times greater. It is therefore evident that eq'n (2) fails to give correct results in the case where i differs from unity but by quantities of the order m^2.

In the lunar theory, however, the disturbing force of the sun is comparatively great in respect to the central force; and as some doubt may exist

as to its competency to completely change the character of the elliptical inequalities due to the central force, we shall now show that the application of the same method to the planetary theories, in which case the disturbing forces are almost infinitesimally small in comparison with the central force, leads to results as extravagant and absurd as the perturbations produced by analysis (though ascribed to gravita'n), in the theories of the moon's motion.

For this purpose we shall consider the motion of Mercury as disturbed by Neptune. If we designate the coordinates of Mercury and Neptune by the same notation which we have employed for the moon and sun, respectively, in the lunar theory, we shall have $\mathrm{m}^2 = m'(a^3 \div a'^3) = 0.0000000000114$ = the disturbing function. And if we neglect the eccentricity and inclination of Neptune's orbit, the perihelion and node of Mercury's orbit would have the following motions on the ecliptic, t being reckoned in Julian years; namely,

$$\delta\omega = -\delta\Omega = +\tfrac{3}{4}\mathrm{m}^2 nt = +0''.0004603\, t.$$

Now according to Mr. Hill's logic, so long as the perihelion and node move at all, equation (2) must give the correct value of the perturbation of the radius vector depending on the argument $nt+\omega-2\Omega$; consequently

$$\delta\frac{1}{r} = -\tfrac{5}{8}e\gamma^2\cos(nt+\omega-2\Omega);$$

and this gives for the corresponding perturbation in longitude

$$\delta v = -\tfrac{5}{4}e\gamma^2\,(=805'')\sin(nt+\omega-2\Omega). \qquad (7)$$

Now the pure elliptic motion of Mercury gives rise to the following inequality in the longitude; namely,

$$\delta v = +\tfrac{1}{2}e\gamma^2\,(=322'')\sin(nt+\omega-2\Omega);$$

and the sum of these two values of δv would be the true value of the equation, in order to allow for the perturbation by Neptune.

Now I find that the greatest perturbation of Mercury by Neptune is that depending on the argum't of the evection in the lunar theory, and amounts to only $0''.0125$, while the coefficient of equation (7) amounts to only $-0''.000000193$; which is altogether more probable than the preceding value.

Suppose now that we have a second disturbing planet, and call the disturbing function m'^2, the motion of the perihelion and node arising in consequence would be $\delta\omega = -\delta\Omega = \tfrac{3}{4}\mathrm{m}'^2 nt$; and the integral of (3) would be

$$\delta\frac{1}{r} = -\tfrac{5}{8}e\gamma^2\left[\frac{\mathrm{m}^2+\mathrm{m}'^2}{\mathrm{m}^2+\mathrm{m}'^2}\right]\cos(nt+\omega-2\Omega) = -\tfrac{5}{8}e\gamma^2\cos(nt+\omega-2\Omega),$$

the same as for a single disturbing planet. It is evident that any number of additional disturbing planets might be considered in the same way, and give the total perturbation by all the planets the same as that arising from a

single one. The perturbations depending on the proposed argument are therefore entirely indeterminate by equation (2).

Now suppose we were to force such an inequality as is given by equation (7) into the theory and tables of Mercury; it is evident that it would give rise to several inequalities of sensible magnitude, dependiag on different arguments; and also that the differences between theory and observation wo'd be very nearly equal to the sum of the inequalities arising from the new equations thus introduced; since the existing tables very closely represent the motion of Mercury. Assuming the theory including these new equat's to be correct, we should endeavor to make the differences between theory and observation disappear by applying corrections to the elements of the orbit. In this way we should be able to make the residuals disappear more or less completely for a time, since the errors of the elements would partially compensate for the errors of the theory and tables. But no amount of tinkering of the elements and theory possessing such an inherent source of error would suffice to produce tables which would permanently represent the motion of Mercury with the precision required by observation.

Now it appears to me that the lunar theory is in just the condition that the theory and tables of Mercury would be in this supposed case. Several inequalities of considerable magnitude which have no existence in nature have been forced in to the theory and tables of the moon; then corrections of the elements have been determined by means of numerous equations of condition, by which means we have neither correct elements nor correct theories; since equations of condition are powerless to correct for constant or systematic sources of error.

It is proper to remark in this connection, that La Place has stated in book II., chapt. V, of the *Me'canique Ce'leste*, where the integral of equation (1) first appears, that the integral takes a different form from equation (2), in the case of N being equal to i. For this supposed case he gives

$$\delta r = -\frac{m^2}{4N^2}\cos(Nt-\varepsilon)-\frac{m^2 t}{2N}\sin(Nt-\varepsilon) \qquad (8)$$

which is easily proved by differentiation to be correct. Now the last term of this equation vanishes at the epoch, when $t=0$; and we need only consider the second term in this connection. Since $N=1-\frac{3}{4}m^2$, and $i=1+\frac{9}{4}m^2$, we shall have $N = 0.9958$ and $i = 1.0126$, in the case of the moon disturbed by the sun. These numbers are not exactly equal, although very nearly so. In the case of Mercury disturbed by Neptune, we have

$$N = 0.99999999999145, \text{ and } i = 1.0000000002565.$$

These numbers approach very nearly to the ratio of equality; but it may very easily be shown that the elements of the orbits should be treated as con-

stant in the differential equations, and we should then have, rigorously, $N = i = 1$; and the integral of equation (1) would become

$$\delta r = -\tfrac{1}{4}m^2\cos(it-\varepsilon).$$

Equation (3) would also become

$$\frac{d^2\delta\frac{1}{r}}{dt^2}+\delta\frac{1}{r}-\tfrac{9}{2}m^2e\gamma^2\cos(nt+\omega-2\Omega) = 0; \tag{9}$$

the integral of which would be

$$\delta\frac{1}{r} = \tfrac{9}{8}m^2e\gamma^2\cos(nt+\omega-2\Omega). \tag{10}$$

This would give for the perturbation of the moon's longitude

$$\delta v = [\tfrac{9}{4}m^2e\gamma^2 = 1''.2]\sin(nt+\omega-2\Omega). \tag{11}$$

The perturbation depending on this argument, therefore, amounts to only 1″.2 instead of 111″ as determined by Ponte´coulaut, Plana and others.

From this examination it is apparent that the integral given by equation (2) is not applicable to those equations in which the coefficient of t in the argument is simply the mean motion of the moon; and had La Place and his successors remembered this circumstance in their calculations, the theory of the moon's motion would have been relieved of its most embarrassing features; its development would have been confined within narrower limits, and its permanent improvement been thereby greatly facilitated.

Cleveland, Aug. 25, 1882.

Postscript.—The preceding article was prepared for the November number of the ANALYST; but as it was not transmitted to the editor until the matter for that number had been selected, it was thought best to add the following as a *postscript* to that article, in order that the whole difficulty with which the lunar theory is embarrassed may be clearly and unmistakably traced to its source,

The equation

$$\left.\begin{matrix}\dfrac{d^2\delta u}{dv^2}+(1-\tfrac{3}{2}\mu)\delta u-\tfrac{21}{4}m^2e\gamma^2\cos(v+\omega-2\Omega)\\ +\tfrac{15}{8}m^2\dfrac{a}{a'}e'\cos(v-\omega')\end{matrix}\right\} = 0, \tag{12}$$

given by Plana on page 72, *tome* II of his *The´orie du Mouvement de la Lune*, is a particular case of the general equation

$$\frac{d^2y}{dt^2}+N^2y+k\cos(it-\varepsilon) = 0, \tag{13}$$

the integral of which is, in the case where $N = i$,

$$y = -\frac{k}{4i^2}\cos(it-\varepsilon)-\frac{kt}{2i}\sin(it-\varepsilon). \tag{14}$$

The integral of equ'n (12), neglecting the last term, should therefore be

$$\delta u = \tfrac{21}{16} m^2 e\gamma^2 \cos(v+\omega-2\Omega) + \tfrac{21}{8} m^2 e\gamma^2\, v \sin(v+\omega-2\Omega), \qquad (15)$$

instead of

$$\delta u = -\tfrac{7}{8} e\gamma^2 \cos(v+\omega-2\Omega), \qquad (16)$$

as given by Plana.

The last term of equation (15) arises from the secular variation of the elements of the moon's orbit; while the first term of the second member arises from their periodic variations. In order to prove this we shall take the values of the secular variation of the elements given by Plana, in *tome* I, pp. 96, 97; namely,

$$\left.\begin{aligned} \delta\gamma &= \tfrac{5}{8} e^2\gamma \cos(2\omega-2\Omega) \\ \gamma\delta\Omega &= \tfrac{5}{8} e^2\gamma \sin(2\omega-2\Omega) \end{aligned}\right\}; \ (17) \qquad \left.\begin{aligned} \delta e &= -\tfrac{7}{8} e\gamma^2 \cos(2\omega-2\Omega) \\ e\delta\omega &= \tfrac{7}{8} e\gamma^2 \sin(2\omega-2\Omega) \end{aligned}\right\}. \ (18)$$

It is evident that the variations of the elements ought to simultaneously vanish at the epoch of the tables; and as equations (17) and (18) do not satisfy that condition, we must add a constant to each of these equations in order to make the variations vanish at a given epoch, when $\omega = \omega_0$, $\Omega = \Omega_0$. Equations (17) and (18) will therefore become

$$\left.\begin{aligned} \delta\gamma &= \tfrac{5}{8} e^2\gamma \left[\cos(2\omega-2\Omega) - \cos(2\omega_0-2\Omega_0)\right] \\ \gamma\delta\Omega &= \tfrac{5}{8} e^2\gamma \left[\sin(2\omega-2\Omega) - \sin(2\omega_0-2\Omega_0)\right] \end{aligned}\right\}; \qquad (19)$$

$$\left.\begin{aligned} \delta e &= -\tfrac{7}{8} e\gamma^2 \left[\cos(2\omega-2\Omega) - \cos(2\omega_0-2\Omega_0)\right] \\ e\delta\omega &= \tfrac{7}{8} e\gamma^2 \left[\sin(2\omega-2\Omega) - \sin(2\omega_0-2\Omega_0)\right] \end{aligned}\right\}. \qquad (20)$$

Now the principal term in the value of u is

$$u = e\cos(v-\omega_0); \qquad (21)$$

and its variation corresponding to small variations of e and ω_0 will be given by the equation

$$\delta u = \left(\frac{du}{de}\right)\delta e + \left(\frac{du}{d\omega}\right)\delta\omega = \cos(v-\omega_0)\delta e + \sin(v-\omega_0) e\delta\omega. \qquad (22)$$

If we substitute the values of δe and $e\delta\omega$ given by eq'ns (20) we shall find

$$\left.\begin{aligned} \delta u &= -\tfrac{7}{8} e\gamma^2 \left[\cos(v-\omega_0+2\omega-2\Omega) - \cos(v+\omega_0-2\Omega_0)\right] \\ &= \tfrac{7}{4} e\gamma^2 \sin(\omega-\Omega-\omega_0+\Omega_0)\sin(v+\omega-\Omega-\Omega_0) \end{aligned}\right\}. \qquad (23)$$

In the case of constant elements we have $\omega = \omega_0$, $\Omega = \Omega_0$, and δu vanishes. But since ω and Ω are variable, we have, according to Plana,

$$\omega = \omega_0 + \tfrac{3}{4} m^2 v, \qquad \Omega = \Omega_0 - \tfrac{3}{4} m^2 v, \qquad (24)$$

so that $\omega-\Omega-\omega_0+\Omega_0 = \tfrac{3}{2} m^2 v$; and as we are retaining only the terms of the order m^2, we may put

$$\sin(\omega-\Omega-\omega_0+\Omega_0) = \omega-\Omega-\omega_0+\Omega_0 = \tfrac{3}{2} m^2 v, \qquad (25)$$

and equation (23) will become

$$\delta u = \tfrac{21}{8} m^2 e\gamma^2\, v \sin(v+\omega-\Omega-\Omega_0 = v+\omega-2\Omega \textit{ at the epoch}). \qquad (26)$$

This is the same as the last term of equation (15), and proves the first p't of the proposition. The second part may be proved in a similar manner by taking the variation of the second term of the value of u and substituting the periodic variation of the elements.

We thus see that the variation of the elements leads to the same results as are derived from the variation of the coordinates, when the proper constants are added to the integrals of these variations.

The integral of equation (12) depending on $\cos(v-\omega')$ is

$$\delta u = -\tfrac{15}{32}m^2(a\div a')\,e'\cos(v-\omega') - \tfrac{15}{16}m^2(a\div a')e'\,v\sin(v-\omega'), \quad (27)$$

instead of $\delta u = \frac{5}{4}(a\div a')\,e'\cos(v-\omega')$, as given by Plana.

The last term of equation (27) arises from the secular variation of e and ω, depending on the distance between the perigees of the sun and moon. To prove this we shall observe that the differential equations of e and ω contain the terms

$$\left.\begin{aligned} \frac{de}{dv} &= -\tfrac{15}{16}m^2\frac{a}{a'}\,e'\sin(\omega-\omega') \\ e\frac{d\omega}{dv} &= -\tfrac{15}{16}m^2\frac{a}{a'}\,e'\cos(\omega-\omega') \end{aligned}\right\}. \quad (28)$$

If we integrate these equations in the same way that Plana has done, and also add a constant to the integrals so that δe and $\delta\omega$ may *simultaneously vanish at the epoch*, we shall find,

$$\left.\begin{aligned} \delta e &= \tfrac{5}{4}(a\div a')\,e'[\cos(\omega-\omega')-\cos(\omega_0-\omega'_0)] \\ e\delta\omega &= -\tfrac{5}{4}(a\div a')\,e'[\sin(\omega-\omega')-\sin(\omega_0-\omega'_0)] \end{aligned}\right\}. \quad (29)$$

If these values be substituted in equation (22), we shall find, after making the necessary reductions,

$$\delta u = -\tfrac{15}{16}m^2(a\div a')\,e'\,v\sin(v-\omega'), \quad (30)$$

as in the last term of equation (27).

The other term of equation (27) may be found in like manner by substituting the proper periodic variations of e and ω in equation (22).

The two cases which we have examined are the most important ones among the equations of the moon's longitude; but there are two terms of considerable importance in the equations of the moon's latitude, which we shall now examine.

The first of these terms of the latitude is given by Plana in *tome* II, page 118, of his work, as follows:

$$\frac{d^2\delta s}{dv^2}+(1+\tfrac{3}{2}m^2)\delta s-\tfrac{15}{4}m^2e^2\gamma\sin(v-2\omega+\Omega)=0. \quad (31)$$

The integral of this equation is

$$\delta s = \tfrac{15}{16}m^2e^2\gamma\sin(v-2\omega+\Omega)-\tfrac{15}{8}m^2e^2\gamma\,v\cos(v-2\omega+\Omega); \quad (32)$$

instead of $\delta s = +\frac{5}{8}e^2\gamma\sin(v-2\omega+\Omega)$, as given by Plana.

The last term of equation (32) is produced by the secular variation of the node and inclination of the moon's orbit, as may be shown in the following manner:—Since the principal term of the latitude is given by the equation

$$s = \gamma \sin (v - \Omega_0), \tag{33}$$

we shall have

$$\delta s = \sin (v - \Omega_0)\delta\gamma - \cos (v - \Omega_0)\gamma\delta\Omega; \tag{34}$$

and if we substitute the values of $\delta\gamma$ and $\gamma\delta\Omega$, which are given by eq. (19) we shall find

$$\left.\begin{array}{l}\delta s = \frac{5}{4}e^2\gamma \sin (\omega_0 - \Omega_0 - \omega + \Omega) \cos (v - \omega_0 - \omega + \Omega) \\ \quad = -\frac{15}{8}m^2e^2\gamma\, v \cos (v - 2\omega + \Omega)\end{array}\right\}, \tag{35}$$

the same as the last term of equation (32).

The second term of the latitude which seems to be entirely erroneous is the one arising from the oblateness of the earth. La Place first computed this inequality; and I am especially desirous of calling attention to it, since the error is common to every theory of the moon's motion which I have examined, including my own. In this examination I shall first follow the method given by La Place in the *Mécanique Céleste*, making use of the translation by Bowditch, since it is more convenient for reference than the original work. I shall then show how the same results may be obtained by the method given in my own work.

If, in the function [5347], *Méc. Cél.*, we substitute for $u \div h^2$ its value $1 \div a^2$, and also put for abridgement

$$m = (a\rho - \tfrac{1}{2}a\varphi)\frac{D^2}{a^2}, \tag{36}$$

it will become

$$2m \sin \varepsilon \cos \varepsilon \sin fv + (g^2 - 1)H \sin fv, \tag{37}$$

ε denoting the obliquity of the ecliptic to the equator.

But in [5350] La Place gives

$$H = -\frac{2m}{g^2 - 1} \sin \varepsilon \cos \varepsilon; \tag{38}$$

consequently the function (37) is *identically equal to nothing.*

Now La Place remarks that the oblateness of the earth *adds* the function (37) to the differential equation of the latitude, by which means it becomes

$$\frac{d^2\delta s}{dv^2} + \delta s + [2m \sin \varepsilon \cos \varepsilon + (g^2 - 1)H] \sin fv = 0; \tag{39}$$

the integral of which he gives as

$$\delta s = -\frac{m}{g - 1} \sin \varepsilon \cos \varepsilon \sin fv. \tag{40}$$

In other words, if *nothing* be *added* to the differential equation of the latitude, its integral will be *increased* by the value of δs in equation (40);—a result which is exceedingly interesting as well as mysterious to a person unacquainted with the wonderful powers of the *calculus*.

But the correct value of the integral of equation (39) is

$$\left.\begin{aligned}\delta s = &-\tfrac{1}{4}[2m \sin \varepsilon \cos \varepsilon + (g^2-1)H] \sin fv \\ &+\tfrac{1}{2}[2m \sin \varepsilon \cos \varepsilon + (g^2-1)H]\, v \cos fv\end{aligned}\right\}, \tag{41}$$

since $f = 1$.

The function H is therefore not correctly given by equation (38), and the function (37) is not equal to *zero*.

In equation (41) the term $-\tfrac{1}{2}m \sin \varepsilon \cos \varepsilon \sin fv$ gives the direct effect of the earth's oblateness on the moon's latitude; while the term $(g^2-1)H \sin fv$ is its indirect action transmitted to the moon by means of the sun but decreased in the ratio of g^2-1 to unity. Now $\tfrac{1}{2}m \sin \varepsilon \cos \varepsilon = 0''.01654 = H$, and $(g^2-1)H = 0''.000133$. Whence it follows that the effect of the earth's oblateness on the moon's latitude is insensible.

The same result may be obtained by the method empl'd in my own work. For the same force which gives rise to the first term of the function (37), produces in the value of $\frac{d\delta_0\theta}{dt}$, eq'n (263) of my Theory of the Moon's Motion, the following terms

$$\frac{d\delta_0\theta}{dt} = mn \sin \varepsilon \cos \varepsilon \left[\tfrac{1}{2} \cos nt - nt \sin nt\right]; \tag{42}$$

and this gives by integration

$$\delta_0\theta = m \sin \varepsilon \cos \varepsilon \left[-\tfrac{1}{2} \sin nt + nt \cos nt\right], \tag{43}$$

which is the same as the corresponding terms of δs in equation (41), since $v = nt$, when we neglect the eccentricity of the moon's orbit.

It is easy to prove that equation (43) is correct, by means of the variation of the elements. For if we add a constant to the secular terms of these variations, which are given by equations (630) and (631) of my work, so that they may simultaneously vanish at the epoch, they will become

$$\left.\begin{aligned}\gamma\delta\Omega &= m \sin \varepsilon \cos \varepsilon \left[\tfrac{1}{2} \sin (2nt-\Omega) - (n \div a') (\sin \Omega - \sin \Omega_0)\right] \\ \delta\gamma &= m \sin \varepsilon \cos \varepsilon \left[\tfrac{1}{2} \cos (2nt-\Omega) + (n \div a') (\cos \Omega - \cos \Omega_0)\right]\end{aligned}\right\}. \tag{44}$$

And if we substitute these values in the equation

$$\delta\theta = \sin (nt-\Omega_0)\delta\gamma - \cos (nt-\Omega_0)\gamma\delta\Omega, \tag{45}$$

the secular terms will produce the following term,

$$\delta\theta = 2\frac{mn}{a'} \sin \varepsilon \cos \varepsilon \sin \tfrac{1}{2}(\Omega-\Omega_0) \cos nt. \tag{46}$$

But since $\Omega-\Omega_0 = a't$, we have $\sin \tfrac{1}{2}(\Omega-\Omega_0) = \tfrac{1}{2}a't$, and eq. (46) becomes

$$\delta\theta = m \sin \varepsilon \cos \varepsilon\, nt \cos nt; \tag{47}$$

which is the same as the last term of (43).

If we now substitute the periodic terms of equations (44) in equation (45) we get the first term of equation (43).

Finally La Place remarks, *Me'c. Ce'l.*, [5398], that the inequality of the moon's motion in latitude arising from the oblateness of the earth is only

the reaction of the nutation of the earth's axis, discovered by Bradley;— a statement which seems strangely at variance with mechanical principles, since the one has a period of only twenty-seven days, while the other has a period of nearly nineteen years.

It is evident that this change in the equation of the moon's latitude affects the corresponding equation of the longitude; and I find that it changes its value from $\delta v = 6\frac{mn}{a'}\gamma \sin \varepsilon \cos \varepsilon \sin \Omega$, to $\delta v = \frac{13}{2}\frac{mn}{a'}\gamma \sin \varepsilon \cos \varepsilon \sin \Omega$. The equations of the moon's motion arising from the oblateness of the earth are, therefore

$$\delta v = 4''.814 \sin \Omega, \qquad \delta\theta = -0''.0165 \sin nt, \tag{48}$$

instead of

$$\delta v = 4''.444 \sin \Omega, \qquad \delta\theta = -8''.226 \sin nt,$$

previously found.

But there is one point in Mr. Hill's "Review" which seems to be well made out, and for which I wish to thus publicly tender him my grateful acknowledgements. And that is in relation to the reduction of the moon's longitude from the plane of its orbit to the ecliptic, and which amounts to the correction which I have now applied to the equation of the longitude arising from the oblateness of the earth.

I have thus shown with considerable detail, not only that the lunar theories of my predecessors are erroneous by terms of the third order, but have also shown just how the errors were introduced into the several theories. The errors were first committed by La Place; and his methods of integration were blindly followed by his successors until his results came to be accepted by astronomers as the legitimate effects of the law of universal gravitation. Had La Place been a less skilful mathematician it is not probable that he would have made such mistakes; since he would have been content to consider one power of the disturbing force at a time, and he wo'd then have been obliged to regard the elements as constant in the integrat'ns; and then, in all the cases which I have pointed out, he would have found *zero* for a divisor and *infinity* for the perturbations. And such results could not have failed to indicate their origin in the improper application of a general formula of integration to a special case in which such formula fails to give correct results. I therefore confidently believe that existing theories when corrected for the sources of error which I have pointed out; together with a careful revision of the elem'ts of the lunar orbit, will give the moon's place in the heavens with all the precision required by observation, without carrying the approximat'n to terms of a higher order than have been already calculated.

Cleveland, Oct. 10, 1882.